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LETTER TO THE EDITOR

Traces of products of angular momentum operators

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Abstract. A symmetrical and computationally advantageous analytic expression for $\mathrm{Tr}(J_-^k J_z^l J_+^k)$ is obtained.

On account of applications in many branches of physics, several authors have considered the problem of the computation of traces of products of angular momentum operators in both cartesian and spherical bases (Ambler *et al* 1962a, b, Rose 1962, Witschel 1971, 1975, Subramanian and Devanathan 1974, De Meyer and Vanden Berghe 1978a, b). All such traces can be expressed as linear combinations of traces of products of the form $J_-^k J_z^l J_+^k$. The most recent attempts by De Meyer and Vanden Berghe concern such traces. In their first paper, they obtained an explicit expression for such traces, which unfortunately has two disadvantages. Firstly, even for reasonably large values of l, it becomes quite time-consuming to utilise their expression. Also the expression does not exhibit a natural symmetry under the operation $j \to -j-1$. In their second paper, De Meyer and Vanden Berghe (1978b) obtain recursion relations for these traces which do exhibit the symmetry mentioned above, and the computation of the traces is also relatively simpler. However, the recursion relations are different for even and odd values of l. For even values of l, the computation of the trace is very much more involved.

In this Letter, we have obtained an explicit expression for $\mathrm{Tr}(J_-^k J_z^l J_+^k)$ which exhibits the symmetry $j \to -j-1$ and is very much simpler to use for actual computation of analytical expressions for these traces. In addition, the proof, except for a non-trivial step, is very simple and straightforward.

We write

$$\mathrm{Tr}(k, l) = \mathrm{Tr}(J_-^k J_Z^l J_+^k) \tag{1}$$

and use the usual representation

$$\begin{aligned} J_\pm|jm\rangle &= [(j \mp m)(j \pm m + 1)]^{1/2}|jm \pm 1\rangle \\ J_z|jm\rangle &= m|jm\rangle \end{aligned} \tag{2}$$

to obtain

$$\begin{aligned} \mathrm{Tr}(k, l) &= \mathrm{Tr}(J_-^k J_z^l J_+^k) = \mathrm{Tr}(J_+^k J_-^k J_z^l) \\ &= \sum_m \langle jm|J_+^k J_-^k J_l^k|jm\rangle \\ &= \sum_m m^l (j+m)!(j-m+k)!/(j+m-k)!(j-m)! \end{aligned} \tag{3}$$

where the range of summation is indicated by the non-negativity of the arguments of all the factorials present (this convention is followed in all the equations in which the range of summation is omitted). Using the difference operator ∇, defined by

$$\nabla f(x) = f(x) - f(x-1) \tag{4}$$

we can write

$$\begin{aligned} m^l &= (1-\nabla)^{j-m}((j)^l) \\ &= \sum_t [(j-m)!/t!(j-m-t)!](-1)^t \nabla^t((j)^l). \end{aligned} \tag{5}$$

Substituting for m^l from equation (5) in equation (3) above, and performing the m summation, we arrive at

$$\mathrm{Tr}(k, l) = (2j+k+1)!\,k!\sum_t (-1)^t \frac{(k+t)!}{t!\,(2j-k-t)!\,(2k+1+t)!}\nabla^t((j)^l). \tag{6}$$

In particular,

$$\mathrm{Tr}(k, 0) = (2j+k+1)!\,(k!)^2/(2j-k)!\,(2k+1)!. \tag{7}$$

Thus

$$\mathrm{Tr}(k, l) = \mathrm{Tr}(k, 0)\,{}_2F_1(-2j+k, k+1; 2k+2; \nabla)((j)^l). \tag{8}$$

In the above equation, the hypergeometric function represents a formal *finite* series in the operator ∇.

Using the *quadratic transformation*

$${}_2F_1(a, b; 2b; z) + (1-z)^{-a/2}\,{}_2F_1(a/2, b-a/2; b+\tfrac{1}{2}; -z^2/4(1-z)) \tag{9}$$

(Magnus *et al* 1966) and

$$(1-\nabla)^{j-\frac{1}{2}k-t}((j)^l) = (h/2+t)^l \tag{10}$$

we arrive at†

$$\begin{aligned} \mathrm{Tr}(k, l) &= \mathrm{Tr}(k, 0)[(j-k/2)!\,(k+\tfrac{1}{2})!/(j+k/2)!] \\ &\quad \times \sum_{t=0}^{[l/2]} [(j+k/2+t)!/t!\,(j-k/2-t)!\,(k+\tfrac{1}{2}+t)!](\tfrac{1}{4}\nabla^2)^t (k/2+t)^l \end{aligned} \tag{11}$$

where $[l/2]$ expresses the integral part of $l/2$. The above is the analytic expression for $\mathrm{Tr}(k, l)$ which we were seeking. We wish to make a few remarks about this expression.

(i) The symmetry $j \to -j-1$ is built in.

(ii) Each term in the summation factorises into a j-dependent and l-independent part and a j-independent and an l-dependent part. Thus the computation of $\mathrm{Tr}(k, l)$ for many values of l will involve only a few j-dependent coefficients.

(iii) Since

$$\nabla^{2t}((k/2+t)^l) = \sum_n [(2t)!/(t+n)!\,(t-n)!](k/2+n)^l(-1)^{n+t}. \tag{12}$$

$$\nabla^{2t}((k/2+t)^l)_{k=0} = \sum_n [(2t)!/(t+n)!\,(t-n)!]n^l(-1)^{n+t} = 0 \qquad l = \text{odd}.$$

† We are using the notation $\alpha!$ for $\Gamma(\alpha+1)$ even when α is not necessary a non-negative integer.

Thus Tr(k, l) will have k Tr(k, 0) as a factor for odd l.

(iv) $$\mathrm{Tr}(k, l) = \mathrm{Tr}(k, 0) P_{[l/2]}(j(j+1)) \tag{13}$$

where P is a polynomial in $j(j+1)$ of degree $[l/2]$ with coefficients which are rational polynomials in k.

(v) Since

$$\mathrm{Tr}(k, 0) = (2j+1) P_k(j(j+1)) \tag{14}$$

(see equation (7) above),

$$\mathrm{Tr}(k, l) = (2j+1) P_{k+[l/2]}(j(j+1)). \tag{15}$$

This qualitative result was obtained by Subramanian and Devanathan (1974) and has recently been rederived by Kaplan and Zia (1979) using different methods.

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